# Double field formulation of Yang-Mills theory

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**Abstract**

Based on our previous work on the differential geometry for the closed string double field theory, we construct a Yang-Mills action which is covariant under $\mathbf{O}(D,D)$ T-duality rotation and invariant under three-types of gauge transformations: non-Abelian Yang-Mills, diffeomorphism and one-form gauge symmetries. In double field formulation, in a manifestly covariant manner our action couples a single $\mathbf{O}(D,D)$ vector potential to the closed string double field theory. In terms of undoubled component fields, it couples a usual Yang-Mills gauge field to an additional one-form field and also to the closed string background fields which consist of a dilaton, graviton and a two-form gauge field. Our resulting action resembles a twisted Yang-Mills action.

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# Contents



# 1 Introduction

The low energy effective action for a closed string massless sector takes the following well-known form:

$$S_{\text{eff.}}=\int \mathrm{d}x^D\sqrt{-g}e^{-2\phi}\left[\,R_g+4(\partial\phi)^2-\tfrac{1}{12}H^2\,\right]\,, \tag{1.1}$$

where $g_{\mu\nu}$ is the $D$-dimensional spacetime metric with its scalar curvature, $R_g$; $\phi$ is the string theory dilaton; and $H$ is the three form field strength of a two form gauge field, $B_{\mu\nu}$. In a double field theory (DFT) formalism developed by Hull *et all*, in [1–4], the above action was reformulated as

$$\begin{aligned}S_{\text{DFT}}=\int \mathrm{d}y^{2D}\,e^{-2d}\Big[\;&\mathcal{H}^{AB}\left(4\partial_A\partial_Bd-4\partial_Ad\partial_Bd+\tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD}-\tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)\\&+4\partial_A\mathcal{H}^{AB}\partial_Bd-\partial_A\partial_B\mathcal{H}^{AB}\;\Big]\,.\end{aligned} \tag{1.2}$$

Herein the spacetime dimension is formally doubled from $D$ to $2D$ with coordinates $x^\mu\rightarrow y^A=(\tilde{x}_\mu,x^\nu)$; $d$ denotes the double field theory 'dilaton' given by $e^{-2d}=\sqrt{-g}e^{-2\phi}$; and $\mathcal{H}_{AB}$ is a $2D\times 2D$ matrix of the form,

$$\mathcal{H}_{AB}=\begin{pmatrix}g^{\mu\nu} & -g^{\mu\kappa}B_{\kappa\sigma}\\ B_{\rho\kappa}g^{\kappa\nu} & g_{\rho\sigma}-B_{\rho\kappa}g^{\kappa\lambda}B_{\lambda\sigma}\end{pmatrix}\,. \tag{1.3}$$

All the spacetime indices, $A, B, C, \cdots$ , are $2D$-dimensional vector indices which can be raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric, $\eta$,

$$\eta := \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) . \tag{1.4}$$

As a field theory counterpart of the level matching condition in closed string theory, it is required that,[1] all the fields in double field theory as well as all of their possible products should be annihilated by the $\mathbf{O}(D,D)$ d'Alembert operator, $\partial^2 = \partial_A \partial^A$,

$$\partial^2 \Phi \equiv 0 \,, \qquad\qquad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0 \,. \tag{1.5}$$

This constraint, which one may call 'the level matching constraint', actually means that the theory is not truly doubled: there is a choice of coordinates $(\tilde{x}', x')$, related to the original coordinates $(\tilde{x}, x)$, by an $\mathbf{O}(D,D)$ rotation, in which all the fields do not depend on the $\tilde{x}'$ coordinates [3]. Remarkably, while the double field theory action, $S_{\rm DFT}$ (1.2), reduces to the effective action, $S_{\rm eff.}$ (1.1), upon the level matching constraint, the double field theory formulation manifests the $\mathbf{O}(D,D)$ covariance of the action[2] and hence the T-duality first noted by Buscher [5–7] and further studied in [9–16].

However, what is not obvious about the above DFT action (1.2) is that it possesses gauge symmetry, which must be the case [4, 17], since restricted on the $x$-hyperplane the action (1.2) is nothing but a rewriting of the effective action (1.1) while the latter surely enjoys both the $D$-dimensional diffeomorphism, $x^\mu \rightarrow x^\mu + \delta x^\mu$, and the gauge symmetry of the two form field, $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$. That is to say, in contrast to the effective action (1.1) where the gauge symmetry is manifest yet T-duality is not, in the DFT action given in the form (1.2) it is quite the opposite.

In order to manifest both the $\mathbf{O}(D,D)$ structure and the gauge symmetry, in our previous work [18], we conceived a differential geometry characterized by a *projection* satisfying the following defining properties,

$$P_A{}^B P_B{}^C = P_A{}^C \,, \qquad\qquad P_{AB} = P_{BA} \,. \tag{1.6}$$

Further demanding that the upper left $D \times D$ block of $2P - 1$ is non-degenerate, the projection is related to the matrix, $\mathcal{H}_{AB}$ (1.3), by

$$P_A{}^B = \tfrac{1}{2}(\delta_A{}^B + \mathcal{H}_A{}^B) \,. \tag{1.7}$$

[1] Note that throughout our paper, the equivalence symbol, '$\equiv$', denotes the equality up to the level matching constraint (1.5).

[2] Without imposing the level matching constraint, the $\mathbf{O}(D,D)$ transformation surely corresponds to a Noether symmetry of the $2D$-dimensional field theory. After imposing the constraint, the double field theory is, by nature, $D$-dimensional: it lives on a $D$-dimensional hyperplane. As the $\mathbf{O}(D,D)$ transformation then rotates the entire hyperplane, the $\mathbf{O}(D,D)$ rotation acts *a priori* as a duality rather than a Noether symmetry of the $D$-dimensional theory. After further dimensional reductions, it becomes a Noether symmetry of the reduced action, as verified by Buscher [5–7] (*c.f.* [8]).

In terms of a certain differential operator compatible with the projection – which we review later – we were able to identify the underlying differential geometry of the double field theory and, in particular, to rewrite the original DFT action (1.2) in a compact manner,[3]

$$S_{\mathrm{DFT}} = \int \mathrm{d}y^{2D}\, e^{-2d}\, \mathcal{H}^{AB}\left(4\nabla_A d\, \nabla_B d + S_{AB}\right) . \tag{1.8}$$

In this paper, we apply our differential geometric tools in [18] to Yang-Mills theory with an arbitrary gauge group, $\mathbf{G}$. We construct a Yang-Mills action which is covariant under the $\mathbf{O}(D,D)$ rotation and invariant under three-types of gauge transformations: non-Abelian Yang-Mills, diffeomorphism and one-form gauge symmetries. The latter two amount to the DFT gauge symmetry, as summarized below:

- $\mathbf{O}(D,D)$ T-duality
- Gauge symmetry $\left\{\begin{array}{l}\text{Yang-Mills gauge symmetry}\\[1em] \text{DFT gauge symmetry} \quad \left\{\begin{array}{l}\text{Diffeomorphism}\\ \text{One-form gauge symmetry for } B_{\mu\nu}\end{array}\right.\end{array}\right.$

In double field formulation, our action couples a single $\mathbf{O}(D,D)$ vector potential to the closed string double field theory (1.8), keeping the $\mathbf{O}(D,D)$ T-duality and all the gauge symmetries manifest. In terms of undoubled component fields, the T-duality works in a nontrivial way and the action couples a usual Yang-Mills gauge field, $A_\mu$, to an additional one-form field, $\phi_\mu$, and also to the closed string background fields which consist of the dilaton, graviton and the two-form gauge field, $\phi$, $g_{\mu\nu}$, $B_{\mu\nu}$.

In section 2, we review our previous work [18] on the differential geometry for the closed string double field theory, and set up our notations. In section 3, we present our $\mathbf{O}(D,D)$ covariant Yang-Mills theory, both in the double field formulation (subsection 3.1) and also in terms of undoubled component fields (subsection 3.2). We conclude with some comments in section 4.

[3] Shortly after our work [18], an alternative approach to the underlying differential geometry of the double field theory was proposed by Hohm and Kwak [19] based on earlier works by Siegel [12, 13]. It differs from our approach, as it postulates a covariant derivative whose connection is not *a priori* a physical variable of the double field theory.

## 2 Differential geometry compatible with a projection: *review*

In double field theory, the usual definition of Lie derivative is generalized to [4, 16, 18]

$$\widetilde{\mathcal{L}}_X T_{A_1A_2\cdots A_n} := X^B\partial_B T_{A_1A_2\cdots A_n} + \omega\partial_B X^B T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n} 2\partial_{[A_i}X_{B]}T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,, \tag{2.1}$$

where $\omega$ is the weight of each field, $T_{A_1A_2\cdots A_n}$, and $X^A$ is a local gauge parameter, of which half corresponds to the $D$-dimensional diffeomorphism parameter, $\delta x^\mu$, and the other half matches the one-form gauge symmetry parameter, $\Lambda_\nu$. Up to the level matching constraint (1.5), the commutator of them is closed by the **c**-bracket introduced by Siegel [12],[4]

$$[\widetilde{\mathcal{L}}_X, \widetilde{\mathcal{L}}_Y] \equiv \widetilde{\mathcal{L}}_{[X,Y]_{\mathbf{C}}}\,, \qquad [X,Y]^A_{\mathbf{C}} = X^B\partial_B Y^A - Y^B\partial_B X^A + \tfrac{1}{2}Y^B\partial^A X_B - \tfrac{1}{2}X^B\partial^A Y_B\,. \tag{2.2}$$

By definition in double field theory, *covariant* tensors ($\omega = 0$) or tensor densities follow the gauge transformation rule dictated by the generalized Lie derivative,

$$\delta_X T_{A_1A_2\cdots A_n} = \widetilde{\mathcal{L}}_X T_{A_1A_2\cdots A_n}\,. \tag{2.3}$$

Examples include for a tensor, $\mathcal{H}_{AB}$, and for a scalar density with weight one, $e^{-2d}$, such that[5]

$$\begin{aligned} \delta_X\mathcal{H}_{AB} &= \widetilde{\mathcal{L}}_X\mathcal{H}_{AB} &&= X^C\partial_C\mathcal{H}_{AB} + (\partial_A X_C - \partial_C X_A)\mathcal{H}^C{}_B + (\partial_B X_C - \partial_C X_B)\mathcal{H}_A{}^C\,,\\ \delta_X\left(e^{-2d}\right) &= \widetilde{\mathcal{L}}_X\left(e^{-2d}\right) &&= \partial_A\left(X^A e^{-2d}\right)\,. \end{aligned} \tag{2.4}$$

The latter suggests, with $\widetilde{\mathcal{L}}_X\left(e^{-2d}\right) = -2(\widetilde{\mathcal{L}}_X d)e^{-2d}$,

$$\delta_X d = \widetilde{\mathcal{L}}_X d := X^A\partial_A d - \tfrac{1}{2}\partial_B X^B\,. \tag{2.5}$$

The DFT action (1.2) is indeed invariant under the above gauge transformation (2.4), as first shown in [4].

In our previous work [18], we introduced the following *projection-compatible derivative*, $\nabla_C$, which acts on tensors, tensor densities as well as their derivative-descendants as

$$\nabla_C T_{A_1A_2\cdots A_n} = \partial_C T_{A_1A_2\cdots A_n} - \omega\Gamma^B{}_{BC}T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n}\Gamma_{CA_i}{}^{B}T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,, \tag{2.6}$$

where the connection is, with the projection, (1.6), (1.7), and its complementary projection, $\bar{P} := 1 - P$, given by

$$\Gamma_{CAB} := 2P_{[A}{}^D\bar{P}_{B]}{}^E\partial_C P_{DE} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC}\,. \tag{2.7}$$

[4]Upon the level matching constraints the **c**-bracket itself reduces to the Courant bracket [20], as recognized in [2].

[5]Another example of a covariant tensor is the **c**-bracket of two covariant vectors, $\delta_X\left([X,Y]^A_{\mathbf{C}}\right) \equiv \widetilde{\mathcal{L}}_X\left([X,Y]^A_{\mathbf{C}}\right)$ [21].

This connection was uniquely determined in terms of the projections and their derivatives,[6] by requiring

$$\nabla_A \eta_{BC} = 0\,, \qquad\qquad \nabla_A P_{BC} = 0\,, \tag{2.8}$$

and

$$\Gamma_{CAB} + \Gamma_{CBA} = 0\,, \qquad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0\,. \tag{2.9}$$

The unique feature of the projection-compatible derivative is that, acting on a covariant tensor, although it does not lead to a covariant quantity,

$$\left(\delta_X - \widetilde{\mathcal{L}}_X\right) \nabla_C T_{A_1 A_2 \cdots A_n} \equiv 2 \sum_{i=1}^{n} \left(P_{A_i}{}^D P_B{}^E P_C{}^F + \bar{P}_{A_i}{}^D \bar{P}_B{}^E \bar{P}_C{}^F\right) \partial_F \partial_{[D} X_{E]} T_{A_1 \cdots A_{i-1}}{}^{B}{}_{A_{i+1} \cdots A_n}\,, \tag{2.10}$$

after being contracted properly with the projections, it can be covariantized as

$$\begin{aligned} &\left(\delta_X - \widetilde{\mathcal{L}}_X\right)\left(P_C{}^D \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n}\right) \equiv 0\,, \\ &\left(\delta_X - \widetilde{\mathcal{L}}_X\right)\left(\bar{P}_C{}^D P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n}\right) \equiv 0\,. \end{aligned} \tag{2.11}$$

Thanks to the symmetric properties (2.9), all the ordinary derivatives in the definitions of the generalized Lie derivative (2.1) and the $\mathbf{c}$-bracket (2.2) can be replaced by our projection-compatible derivatives,[7]

$$\begin{aligned} \widetilde{\mathcal{L}}_X T_{A_1 \cdots A_n} &= X^B \nabla_B T_{A_1 \cdots A_n} + \omega \nabla_B X^B T_{A_1 \cdots A_n} + \textstyle\sum_{i=1}^{n} 2 \nabla_{[A_i} X_{B]} T_{A_1 \cdots A_{i-1}}{}^{B}{}_{A_{i+1} \cdots A_n}\,, \\ [X, Y]^A_{\mathbf{C}} &= X^B \nabla_B Y^A - Y^B \nabla_B X^A + \tfrac{1}{2} Y^B \nabla^A X_B - \tfrac{1}{2} X^B \nabla^A Y_B\,. \end{aligned} \tag{2.12}$$

Postulating this property to hold also for the gauge transformation of the dilaton (2.5), and writing

$$\nabla_A(e^{-2d}) = (-2\nabla_A d) e^{-2d}\,, \qquad \nabla_A \nabla_B (e^{-2d}) = (-2\nabla_A \nabla_B d + 4\nabla_A d \nabla_B d) e^{-2d}\,, \tag{2.13}$$

it is natural further to set, as if $\nabla_A d$ has trivial weight,

$$\nabla_A d := \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA}\,, \qquad \nabla_A \nabla_B d := \partial_A \nabla_B d + \Gamma_{AB}{}^C \nabla_C d\,. \tag{2.14}$$

[6]One possible generalization of (2.7) which we have not taken seriously is to include the dilaton and its derivative in the connection,

$$\Gamma_{CAB} \rightarrow \Gamma'_{CAB} := \Gamma_{CAB} - \tfrac{2}{D-1}(P_{CA} P_{BD} - P_{CB} P_{AD} + \bar{P}_{CA} \bar{P}_{BD} - \bar{P}_{CB} \bar{P}_{AD}) \nabla^D d\,.$$

The resulting derivative satisfies (2.8), (2.9), (2.12) and further that $\nabla' d = \partial_A d + \frac{1}{2}\Gamma'^B{}_{BA} = 0$, whilst it does not affect the covariant quantities in (2.11). However, it becomes singular in the case of $D = 1$.

[7]The weight of a gauge symmetry parameter is taken to be zero, such that $\nabla_A X^B = \partial_A X^B + \Gamma_A{}^B{}_C X^C$.

Now, with the curvature defined in standard way,

$$R_{CDAB} := \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E} \Gamma_{BED} - \Gamma_{BC}{}^{E} \Gamma_{AED} \,, \tag{2.15}$$

if we set

$$S_{ABCD} := \tfrac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB} \Gamma_{ECD} \right) \,, \qquad S_{AB} := S^{C}{}_{ACB} \,, \tag{2.16}$$

the following quantities are all gauge covariant [18],

$$\mathcal{R}_{AB} := P_A{}^{C} \bar{P}_B{}^{D} \left( S_{CD} + 2\nabla_{(C} \nabla_{D)} d \right) \,, \tag{2.17}$$

$$\mathcal{R} := \mathcal{H}^{AB} \left( 4\nabla_A \nabla_B d - 4\nabla_A d \, \nabla_B d + S_{AB} \right) \,, \tag{2.18}$$

$$P^{AB} (\nabla_A - 2\nabla_A d) V_B \,, \tag{2.19}$$

$$\bar{P}^{AB} (\nabla_A - 2\nabla_A d) V_B \,, \tag{2.20}$$

$$P^{AB} \bar{P}_{C_1}{}^{D_1} \cdots \bar{P}_{C_n}{}^{D_n} \left[ \nabla_A \nabla_B T_{D_1 \cdots D_n} - 2(\nabla_A d) \nabla_B T_{D_1 \cdots D_n} \right] \,, \tag{2.21}$$

$$\bar{P}^{AB} P_{C_1}{}^{D_1} \cdots P_{C_n}{}^{D_n} \left[ \nabla_A \nabla_B T_{D_1 \cdots D_n} - 2(\nabla_A d) \nabla_B T_{D_1 \cdots D_n} \right] \,, \tag{2.22}$$

in addition to the ones in (2.11),[8]

$$\begin{aligned} & P_C{}^{D} \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n} \,, \\ & \bar{P}_C{}^{D} P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n} \,. \end{aligned} \tag{2.23}$$

As a matter of fact, up to a surface term, the double field theory Lagrangian in (1.8) is equivalent to $e^{-2d}\mathcal{R}$, while its equations of motion for the dilaton and the projection are $\mathcal{R} = 0$ and $\mathcal{R}_{(AB)} = 0$ respectively.

Some useful identities to note are

$$S_{ABCD} = S_{[AB][CD]} \,, \qquad S_{ABCD} = S_{CDAB} \,, \qquad S_{A[BCD]} = 0 \,, \tag{2.24}$$

$$P_A{}^{E} \bar{P}_B{}^{F} P_C{}^{G} \bar{P}_D{}^{H} S_{EFGH} \equiv 0 \,, \qquad P_A{}^{E} P_B{}^{F} \bar{P}_C{}^{G} \bar{P}_D{}^{H} S_{EFGH} \equiv 0 \,, \tag{2.25}$$

$$4\nabla_A \nabla^A d - 4\nabla_A d \, \nabla^A d + S \equiv 0 \,. \tag{2.26}$$

[8] Successive application of (2.23) with more than one covariant vectors also leads to the following gauge covariant higher order derivatives:

$$\begin{aligned} & \left( \textstyle\prod_{i=1}^{m} V_i^B P_B{}^C \nabla_C \right) \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \cdots \bar{P}_{A_n}{}^{B_n} T_{B_1 B_2 \cdots B_n} \,, \\ & \left( \textstyle\prod_{i=1}^{m} V_i^B \bar{P}_B{}^C \nabla_C \right) P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \cdots P_{A_n}{}^{B_n} T_{B_1 B_2 \cdots B_n} \,. \end{aligned}$$

Under an arbitrary infinitesimal transformation of the projection satisfying

$$\delta P = P\delta P\bar{P} + \bar{P}\delta PP\,, \tag{2.27}$$

the connection and $S_{ABCD}$ transform as

$$\begin{array}{ll}
\delta\Gamma_{CAB} = & 2P_{[A}{}^{D}\bar{P}_{B]}{}^{E}\nabla_{C}\delta P_{DE} + 2(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E})\nabla_{D}\delta P_{EC} \\
 & -\,\Gamma_{FDE}\,\delta(P_{C}{}^{F}P_{A}{}^{D}P_{B}{}^{E} + \bar{P}_{C}{}^{F}\bar{P}_{A}{}^{D}\bar{P}_{B}{}^{E})\,, \\
\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,. &
\end{array} \tag{2.28}$$

# 3 $\mathbf{O}(D,D)$ covariant Yang-Mills theory

## 3.1 Double field formulation

Our main result in the present paper comes from generalizing the previous analysis on the covariant quantities, especially (2.23), to Yang-Mills theory with a generic non-Abelian gauge group, $\mathbf{G}$. We postulate a DFT vector potential, $V_A$, which is in the adjoint representation of the Lie algebra of the gauge group, $\mathcal{G}$. For a DFT tensor, $T_{A_1A_2\cdots A_n}$ which is in the fundamental representation of $\mathcal{G}$, we define with the projection-compatible derivative (2.6),

$$\mathcal{D}_C T_{A_1A_2\cdots A_n} := \nabla_C T_{A_1A_2\cdots A_n} - iV_C T_{A_1A_2\cdots A_n}\,. \tag{3.1}$$

This derivative is covariant with respect to the usual Yang-Mills gauge symmetry: with $\mathbf{g}\in\mathbf{G}$, under

$$\begin{array}{lll}
T_{A_1A_2\cdots A_n} & \longrightarrow & \mathbf{g}T_{A_1A_2\cdots A_n}\,, \\
V_A & \longrightarrow & \mathbf{g}V_A\mathbf{g}^{-1} - i(\partial_A\mathbf{g})\mathbf{g}^{-1}\,,
\end{array} \tag{3.2}$$

the derivative transforms in a covariant fashion,

$$\mathcal{D}_C T_{A_1A_2\cdots A_n} \;\longrightarrow\; \mathbf{g}\mathcal{D}_C T_{A_1A_2\cdots A_n}\,. \tag{3.3}$$

Note that the projection and the dilaton are all Yang-Mills gauge singlets such that the projection-compatible derivative (2.6) does not change under the Yang-Mills gauge transformation.

The commutator of the above derivatives reads

$$[\mathcal{D}_A,\mathcal{D}_B]T_{C_1C_2\cdots C_n} = -iF_{AB}T_{C_1C_2\cdots C_n} - \Gamma^{D}{}_{AB}\mathcal{D}_D T_{C_1C_2\cdots C_n} + \sum_{i=1}^{n} R_{C_iDAB}\,T_{C_1\cdots C_{i-1}}{}^{D}{}_{C_{i+1}\cdots C_n}\,, \tag{3.4}$$

where $R_{CDAB}$ is the curvature given in (2.15), and $F_{AB}$ is the field strength of the vector potential,

$$F_{AB} = \partial_A V_B - \partial_B V_A - i\,[V_A, V_B] \ , \tag{3.5}$$

which surely transforms covariantly under the Yang-Mills gauge transformation,

$$F_{AB} \ \longrightarrow \ \mathbf{g} F_{AB} \mathbf{g}^{-1} \,. \tag{3.6}$$

However, this field strength is not DFT gauge covariant,

$$\delta_X F_{AB} \neq \widetilde{\mathcal{L}}_X F_{AB} \,. \tag{3.7}$$

It is necessary to utilize the projection compatible derivative as in (2.23). Hence, instead of (3.5) we consider

$$\mathcal{F}_{AB} := \nabla_A V_B - \nabla_B V_A - i\,[V_A, V_B] = F_{AB} - \Gamma^C{}_{AB} V_C \,. \tag{3.8}$$

Although it is not covariant under the Yang-Mills gauge symmetry,

$$\mathcal{F}_{AB} \ \longrightarrow \ \mathbf{g} \mathcal{F}_{AB} \mathbf{g}^{-1} + i\Gamma^C{}_{AB}(\partial_C \mathbf{g})\mathbf{g}^{-1} \,, \tag{3.9}$$

when its two $\mathbf{O}(D,D)$ vector indices are projected into opposite chiralities,

$$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD} \,, \tag{3.10}$$

it becomes covariant with respect to both the Yang-Mills and the DFT gauge symmetries, thanks to the level matching constraint (1.5) imposed on the explicit expression of the connection (2.7),

$$\begin{array}{lll} P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD} & \longrightarrow & P_A{}^C \bar{P}_B{}^D \mathbf{g} \mathcal{F}_{CD} \mathbf{g}^{-1} \,, \\ \delta_X (P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}) & \equiv & \widetilde{\mathcal{L}}_X (P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}) \,. \end{array} \tag{3.11}$$

Therefore, our double field formulation of a Yang-Mills action is

$$S_{\rm YM} = g_{\rm YM}^{-2} \int \mathrm{d}y^{2D} \ e^{-2d} \, \mathrm{Tr}\big(P^{AB} \bar{P}^{CD} \mathcal{F}_{AC} \mathcal{F}_{BD}\big) \ , \tag{3.12}$$

which can be coupled to the closed string DFT (1.8) as

$$S_{\rm DFT} + S_{\rm YM} = \int \mathrm{d}y^{2D} \ e^{-2d} \left[\mathcal{H}^{AB} \left(4\nabla_A d \, \nabla_B d + S_{AB}\right) + g_{\rm YM}^{-2} \, \mathrm{Tr}\big(P^{AB} \bar{P}^{CD} \mathcal{F}_{AC} \mathcal{F}_{BD}\big)\right] \ . \tag{3.13}$$

These actions are manifestly $\mathbf{O}(D,D)$ covariant, and invariant under both the Yang-Mills and the DFT gauge transformations.

### 3.2 Component field formulation

Here we rewrite the above double field formulation of a Yang-Mills action (3.12) in terms of ordinary undoubled $D$-dimensional component fields, in a similar fashion that the closed string DFT action, $S_{\rm DFT}$ (1.8), reduces to the more familiar looking effective action, $S_{\rm eff.}$ (1.1), upon the level matching constraint.

We first decompose the DFT vector potential into a chiral and an anti-chiral vectors,

$$V_A = V_A^+ + V_A^- \,, \qquad V_A^+ = P_A{}^B V_B \,, \qquad V_A^- = \bar{P}_A{}^B V_B \,, \tag{3.14}$$

such that $\mathcal{H}_A{}^B V_B^{\pm} = \pm V_A^{\pm}$. The chiral and anti-chiral vectors assume the following generic forms,

$$V_A^+ = \tfrac{1}{2}\left(\begin{array}{c} A^{+\lambda} \\ (g{+}B)_{\mu\nu}A^{+\nu} \end{array}\right) , \qquad V_A^- = \tfrac{1}{2}\left(\begin{array}{c} -A^{-\lambda} \\ (g{-}B)_{\mu\nu}A^{-\nu} \end{array}\right) . \tag{3.15}$$

With the field redefinition,

$$A_\mu := \tfrac{1}{2}(A_\mu^+ + A_\mu^-) \,, \qquad \phi_\mu := \tfrac{1}{2}(A_\mu^+ - A_\mu^-) \,, \tag{3.16}$$

which is equivalent to $A_\mu^\pm = A_\mu \pm \phi_\mu$, the DFT vector potential can be parametrized by

$$V_A = \left(\begin{array}{c} \phi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{array}\right) . \tag{3.17}$$

Note that the $D$-dimensional vector indices, $\mu,\nu$, are here and henceforth freely raised or lowered by the $D$-dimensional metric, $g_{\mu\nu}$, in the usual manner.

Direct computation shows, turning off the $\tilde{x}$-dependence,

$$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD} \equiv \tfrac{1}{4}\left(\begin{array}{cc} -\hat{f}^{\lambda\mu} & \hat{f}^{\lambda\tau}(g{+}B)_{\tau\nu} \\ -(g{+}B)_{\rho\sigma}\hat{f}^{\sigma\mu} & (g{+}B)_{\rho\sigma}\hat{f}^{\sigma\tau}(g{+}B)_{\tau\nu} \end{array}\right) , \tag{3.18}$$

where we set

$$\begin{array}{rcl} \hat{f}_{\mu\nu} & := & f_{\mu\nu} - D_\mu\phi_\nu - D_\nu\phi_\mu + i\,[\phi_\mu,\phi_\nu] + H_{\mu\nu\lambda}\phi^\lambda \,, \\ D_\mu\phi_\nu & := & \nabla_\mu\phi_\nu - i\,[A_\mu,\phi_\nu] = \partial_\mu\phi_\nu - \phi_\lambda\gamma^\lambda_{\mu\nu} - i\,[A_\mu,\phi_\nu] \;, \\ f_{\mu\nu} & := & \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \;, \\ H_{\lambda\mu\nu} & := & \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} \,. \end{array} \tag{3.19}$$

Unlike (2.6) and (3.1), in our $D$-dimensional notation, $\nabla_\mu$ denotes the usual diffeomorphism covariant derivative involving the Christoffel symbol, $\gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$, and $D_\mu$ is the diffeomorphism plus Yang-Mills gauge covariant derivative.

It is worth while to note

$$\begin{array}{lll}\hat{f}_{\mu\nu} & = & \nabla_\mu A^-_\nu - \nabla_\nu A^+_\mu - i\left[A^+_\mu, A^-_\nu\right] + H_{\mu\nu\lambda}\phi^\lambda\,, \\ \hat{f}_{[\mu\nu]} & = & f_{\mu\nu} + i\,[\phi_\mu, \phi_\nu] + H_{\mu\nu\lambda}\phi^\lambda\,, \\ \hat{f}_{(\mu\nu)} & = & -(D_\mu\phi_\nu + D_\nu\phi_\mu)\,,\end{array} \tag{3.20}$$

and for (3.18)

$$P_A{}^C\bar{P}_B{}^D\mathcal{F}_{CD} \equiv P_A{}^C\bar{P}_B{}^D \left(\begin{array}{cc} 0 & 0 \\ 0 & \hat{f}_{\mu\nu}\end{array}\right)_{CD}\,. \tag{3.21}$$

Now, from (3.18), it is straightforward to show that the Yang-Mills action in the double field formulation (3.12) reduces to

$$S_{\rm YM} \equiv g_{\rm YM}^{-2}\int \mathrm{d}x^D\,\sqrt{-g}e^{-2\phi}\,\mathrm{Tr}\Big(-\tfrac{1}{4}\hat{f}^{\mu\nu}\hat{f}_{\mu\nu}\Big)\,, \tag{3.22}$$

and hence,

$$S_{\rm DFT} + S_{\rm YM} \equiv \int \mathrm{d}x^D\sqrt{-g}e^{-2\phi}\left[R_g + 4(\partial\phi)^2 - \tfrac{1}{12}H^2 - \tfrac{1}{4}g_{\rm YM}^{-2}\,\mathrm{Tr}\Big(\hat{f}^{\,2}\Big)\right]\,. \tag{3.23}$$

Explicitly, we have for $S_{\rm YM}$ (3.22),

$$\begin{array}{l}\mathrm{Tr}\Big(\hat{f}_{\mu\nu}\hat{f}^{\mu\nu}\Big) = \mathrm{Tr}\Big(\;f_{\mu\nu}f^{\mu\nu} + 2D_\mu\phi_\nu D^\mu\phi^\nu + 2D_\mu\phi_\nu D^\nu\phi^\mu - [\phi_\mu,\phi_\nu][\phi^\mu,\phi^\nu] \\ \qquad\qquad +2if_{\mu\nu}[\phi^\mu,\phi^\nu] + 2\left(f^{\mu\nu} + i[\phi^\mu,\phi^\nu]\right)H_{\mu\nu\sigma}\phi^\sigma + H_{\mu\nu\sigma}H^{\mu\nu}{}_\tau\phi^\sigma\phi^\tau\;\Big)\,.\end{array} \tag{3.24}$$

The above actions (3.22), (3.23) are clearly invariant under both the Yang-Mills and the DFT gauge symmetries. Moreover, though not manifest, by construction it enjoys T-duality.

## 4 Comments

We recall the DFT tensor (3.10) which is fully covariant under the $\mathbf{O}(D,D)$ T-duality as well as all the gauge symmetries, to set

$$\hat{\mathcal{F}}_{AB} := P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}\,. \tag{4.1}$$

Apart from $\mathrm{Tr}(\hat{\mathcal{F}}^{AB}\hat{\mathcal{F}}_{AB})$ which essentially leads to our DFT formulation of the Yang-Mills action (3.12), the following quantity of even power in the field strength is also fully covariant,

$$\mathrm{Tr}\left(\hat{\mathcal{F}}^{A_1B_1}\hat{\mathcal{F}}_{A_2B_1}\hat{\mathcal{F}}^{A_2B_2}\hat{\mathcal{F}}_{A_3B_2}\cdots\hat{\mathcal{F}}^{A_nB_n}\hat{\mathcal{F}}_{A_1B_n}\right)\,. \tag{4.2}$$

Due to the chirality of $\hat{\mathcal{F}}_{AB}$, there is no covariant scalar with odd power. Especially, for the Abelian group,[9] $\mathbf{G} = \mathrm{U}(1)$, we obtain another covariant quantity,[10]

$$\det\!\left(\eta_{AB} + \kappa\,\hat{\mathcal{F}}_{AC}\hat{\mathcal{F}}_B{}^C\right) = \det\!\left(\eta_{AB} + \kappa\,\hat{\mathcal{F}}_{CA}\hat{\mathcal{F}}^C{}_B\right)\,, \tag{4.3}$$

where $\kappa$ is a constant and the determinant is taken over the $\mathbf{O}(D,D)$ vector indices, $A,B$. Since this is a scalar rather than a scalar density, there appears no compulsory reason to take a square root of the determinant constructing a Born-Infeld type action.

In the presence of a curved $D$-brane, string theory can force a topological twisting on a usual Yang-Mills theory, converting scalars into one-form [28]. Especially, when a pure Yang-Mills theory in $(D+D)$-dimensions is reduced to $D$-dimensions, the Lorentz symmetry group coincides with the $R$-symmetry group. If we diagonalize these two, as in topological twisting theories [29–33], we may obtain the following maximally twisted action,

$$S_{\mathrm{twisted}} \equiv -g_{\mathrm{YM}}^{-2}\int \mathrm{d}x^D\,\sqrt{-g}\,\mathrm{Tr}\big(\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \tfrac{1}{2}D_\mu\phi_\nu D^\mu\phi^\nu - \tfrac{1}{4}[\phi_\mu,\phi_\nu][\phi^\mu,\phi^\nu] + \tfrac{1}{2}R_{\mu\nu}\phi^\mu\phi^\nu\big)\,. \tag{4.4}$$

Intriguingly this twisted action resembles our Yang-Mills action (3.22), although they differ in some details.[11] More precise string theory interpretation of our double field formulation of Yang-Mills theory is desirable (for some related works we refer [34–36]). Doubled sigma-model formalism [37–40] may provide useful insights.

[9] Generalization to non-Abelian Born-Infeld action is also doable following various prescriptions, *e.g.* [22–27].

[10] On the other hand, due to the chirality of $\hat{\mathcal{F}}_{AB}$, $\det(\eta_{AB} + \kappa\hat{\mathcal{F}}_{AB})$ is trivial.

[11] To confirm the difference, it is necessary to use the identity,

$$[D_\mu, D_\nu]\phi^\nu + R_{\mu\nu}\phi^\nu + i\,[f_{\mu\nu},\phi^\nu] = 0\,.$$

*Note added*: After submitting the first version of this manuscript to arXiv, a related work by Hohm and Kwak appeared [41]. Their paper attempts the double field theory formulation of the heterotic string effective action, and hence the inclusion of Yang-Mills theories. It is based on an enlarged, yet broken, $\mathbf{O}(D, D+n)$ T-duality, which differs from ours, *i.e.* unbroken $\mathbf{O}(D, D)$.

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